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The converging shock wave from a spherical or cylindrical piston

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A spherical or cylindrical cavity containing quiescent gas begins to contract at high constant radial speed, driving an axisymmetric shock wave inward to collapse at the centre. We analyse the flow field by expanding the solution in powers of time, and calculate 40 terms by delegating the arithmetic to a computer. Analysis of the series for the radius of the shock wave confirms Guderley's local self-similar solution for the focusing, including recent refined values for his similarity exponent, and yields higher terms in his local expansion. In the range of adiabatic exponent where the Guderley solution has been shown not to be unique we find, in accord with a conjecture of Gel'fand, that the smallest admissible similarity exponent is realized.

1. Introduction

The final collapse of an imploding spherical or cylindrical shock wave, as analysed by Guderley (1942), is one of the first examples of a remarkable class of local solutions known in Russian as 'self-similar solutions of the second type' (Zel'dovich & Raizer 1967). In contrast to such familiar self-similar solutions of the 'first type' as the strong point explosion or the boundary layer on a flat plate, the nature of the similarity is revealed neither by dimensional analysis nor by other group properties of the problem, but only by actually solving the equations as a nonlinear eigenvalue problem (Barenblatt 1979). As a consequence, the similarity exponent turns out to be in general an irrational number rather than a simple fraction. Thus Guderley found that the radius of a spherical shock wave in a diatomic gas varies locally as the 0·717-power of the time measured from the instant of collapse.

That exponent has been refined by a succession of subsequent investigators, eventually to 0·717 174 50 by Lazarus & Richtmyer (1977). However, Fujimoto & Mishkin (1978) have recently advanced the unorthodox claim that Guderley's problem can be solved in closed analytic form to yield instead a value of 0·707. This suggestion has been refuted by Lazarus (1980).

Brushlinskii & Kazhdan (1963) report that the solution is unique only if the adiabatic exponent of the gas is less than a critical value, which is 1·87 in the spherical case. Above that value there exists a series of eigenvalues for which the solution is analytic. On the other hand, Welsh (1967) has suggested that the solution is always unique. According to Barenblatt (1979) Gel'fand has conjectured that in reality a solution is always obtained with the smallest similarity exponent in the series, but this awaits proof.

Guderley's local analysis cannot provide the amplitude of his solution. That could be found, and Gel'fand's conjecture confirmed (and the validity of Fujimoto & Mishkin's analysis tested) by solving a global problem, and pursuing the shock wave into its limiting self-similar form at the focus. However, existing numerical solutions (Perry & Kantrowitz 1951; Payne 1957; Berchenko & Korobeinikov 1976) are not sufficiently accurate for that purpose.

We adopt an alternative approach to the global problem, which consists in solving the initial-value problem by expanding the solution in powers of time. This method was introduced by Lee (1968) for a cylindrical piston that collapses with speed proportional to a power of time. He calculated three terms of a double expansion in powers of both time and a parameter representing the departure of the shock wave from infinite strength. Later, Bach & Lee (1969) carried the same kind of calculation to four terms for both cylindrical and spherical waves, using the more complicated initial conditions that the flow is produced by the instantaneous deposition of energy at a finite radius. The first approximation is then the self-similar solution for a strong planar blast wave.

For simplicity, we consider only a spherical or cylindrical piston that collapses with constant inward speed, so that the basic approximation for small time is just the flow produced by impulsive motion of a plane piston; and we assume that the speed is so great that the Mach number of the shock wave is effectively infinite (in Russian terminology, we neglect the counter-pressure, or make the cold-gas approximation), so that we need only a single series. By delegating the mounting arithmetic to a computer we have, in a few minutes, calculated 40 terms of the expansion in powers of time.

We find that the range of convergence of the series varies with the radius, but is nowhere less than the time for the shock wave to collapse onto the axis. Our 40-term expansion therefore describes the whole field accurately up to that instant, except in the immediate vicinity of the collapse. There, by analysing the coefficients, we are able to extract with good accuracy the singular local structure of the flow.

2. Solution expanded in powers of time

We consider a spherical or cylindrical container that is initially of radius R_0 and filled with quiescent perfect gas of density ρ_0 and adiabatic exponent γ. At time $t = 0$ the container begins to contract with a very large constant velocity V, driving ahead of it a shock wave whose radius $R(t)$ is to be found.

The equations of continuity, momentum, and energy are (Sedov 1959, p. 148)

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial r}+j\frac{\rho v}{r} = 0, \tag{1a}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{1b}$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}\right)\frac{p}{\rho^{\gamma}} = 0. \tag{1c}$$

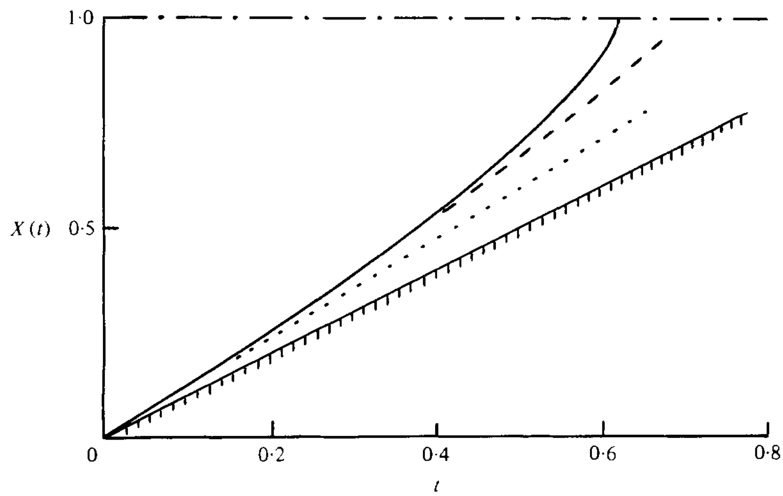

FIGURE 1. History of converging shock wave in (x, t)-plane for spherical piston with $\gamma = \frac{7}{5}$. ⊤⊤⊤⊤⊤⊤, path of piston; ....., 1-term (planar) approximation to shock wave; – – –, 3-term approximation (15); ——, full solution.

Here v is the (outward) radial velocity, and $j = 2$ for a spherical piston and $j = 1$ for a cylindrical one. At infinite Mach number the Rankine–Hugoniot relations give the conditions just behind the shock wave as (Sedov 1959, p. 212)

$$\left.\begin{aligned} v &= \frac{2}{\gamma+1}\dot{R} \\ \rho &= \frac{\gamma+1}{\gamma-1}\rho_0 \\ p &= \frac{2}{\gamma+1}\rho_0\dot{R}^2 \end{aligned}\right\} \quad \text{at} \quad r = R(t). \tag{2}$$

The remaining condition is that of no flow through the piston:

$$v = -V \quad \text{at} \quad r = R_0 - Vt. \tag{3}$$

It is convenient to modify the variables by measuring the distance x inward from the original radius, and correspondingly reversing the sign of the velocity by introducing $u = -v$. Then the history of the flow is represented by the (x, t)-diagram of figure 1. At small time the flow is approximately that produced by a plane piston moving into quiescent gas with speed V. It produces a shock wave moving at constant speed $\frac{1}{2}(\gamma+1)V$. Between the piston and the shock wave the gas has constant speed $u = V$, density $[(\gamma+1)/(\gamma-1)]\rho_0$, and pressure $\frac{1}{2}(\gamma+1)\rho_0 V^2$.

To take advantage of the conical nature of this basic flow, we replace x by a variable

$$\xi = \frac{2}{\gamma-1}\left(\frac{x}{Vt} - 1\right) \tag{4}$$

that varies from zero at the piston to unity at the basic position of the shock wave. Finally, we introduce dimensionless variables by referring lengths to R_0, speed to V, density to ρ_0, pressure to $\rho_0 V^2$, and time to R_0/V. Then the differential equations (1) become

$$[1-(1+\tfrac{1}{2}(\gamma-1)\xi)t]\left[\rho\frac{\partial u}{\partial\xi}+(u-1-\tfrac{1}{2}(\gamma-1)\xi)\frac{\partial\rho}{\partial\xi}+\tfrac{1}{2}(\gamma-1)t\frac{\partial\rho}{\partial t}\right]=\tfrac{1}{2}(\gamma-1)jt\rho u, \quad (5a)$$

$$\rho(u-1-\tfrac{1}{2}(\gamma-1)\xi)\frac{\partial u}{\partial\xi}+\tfrac{1}{2}(\gamma-1)t\rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial\xi}=0, \quad (5b)$$

$$(u-1-\tfrac{1}{2}(\gamma-1)\xi)\left(\rho\frac{\partial p}{\partial\xi}-\gamma p\frac{\partial\rho}{\partial\xi}\right)+\tfrac{1}{2}(\gamma-1)t\left(\rho\frac{\partial p}{\partial t}-\gamma p\frac{\partial\rho}{\partial t}\right)=0, \quad (5c)$$

and the boundary conditions (2), (3) are

$$u=\frac{2}{\gamma+1}\dot{X},\quad \rho=\frac{\gamma+1}{\gamma-1},\quad p=\frac{2}{\gamma+1}\dot{X}^2\quad \text{at}\quad \xi=\frac{2}{\gamma-1}\left[\frac{X(t)}{t}-1\right], \quad (6)$$

$$u=1\quad \text{at}\quad \xi=0. \quad (7)$$

It seems reasonable to assume that the solution is analytic in time. We therefore expand the unknown position of the shock wave in a Taylor series as

$$X(t)=\sum_{n=1}^{\infty}X_n t^n, \quad (8)$$

and likewise expand the flow variables as

$$u=\sum_{n=1}^{\infty}U_n(\xi)t^{n-1},\quad \rho=\sum_{n=1}^{\infty}R_n(\xi)t^{n-1},\quad p=\sum_{n=1}^{\infty}P_n(\xi)t^{n-1}. \quad (9)$$

Here our basic solution gives

$$U_1=1,\quad R_1=\frac{\gamma+1}{\gamma-1},\quad P_1=\tfrac{1}{2}(\gamma+1),\quad X_1=\tfrac{1}{2}(\gamma+1). \quad (10)$$

Substituting into the differential equations (5) and equating like powers of t yields a sequence of triads of first-order linear ordinary differential equations for the coefficients U_n, R_n, P_n, which for the second approximation are

$$\frac{\gamma+1}{\gamma-1}U_2'-\tfrac{1}{2}(\gamma-1)\xi R_2'+\tfrac{1}{2}(\gamma-1)R_2=\tfrac{1}{2}(\gamma+1)j, \quad (11a)$$

$$-\xi U_2'+U_2+\frac{2}{\gamma+1}P_2'=0, \quad (11b)$$

$$\xi(P_2'-\tfrac{1}{2}\gamma(\gamma-1)R_2')-(P_2-\tfrac{1}{2}\gamma(\gamma-1)R_2)=0. \quad (11c)$$

The boundary condition (7) on the piston gives simply $U_n(0)=0$ for all $n>1$. However, the jump conditions (6) are imposed at the unknown position of the shock wave; in order to equate like powers of t we must transfer them to the basic position $\xi=1$ by Taylor-series expansion. This gives for the second approximation

$$U_2(1)=\frac{4}{\gamma+1}X_2,\quad R_2(1)=0,\quad P_2(1)=4X_2. \quad (12)$$

The form of this problem suggests that U_2, R_2, P_2 are linear functions of ξ. Then it is easy to find that

$$\left.\begin{aligned} U_2 &= \frac{\gamma(\gamma-1)}{2(2\gamma-1)}j\xi, \quad R_2 = \frac{\gamma+1}{2\gamma-1}j(1-\xi), \\ P_2 &= \frac{\gamma(\gamma+1)(\gamma-1)}{2(2\gamma-1)}j, \quad X_2 = \frac{\gamma(\gamma+1)(\gamma-1)}{8(2\gamma-1)}j. \end{aligned}\right\} \tag{13}$$

It is clear that in higher approximations the coefficients U_n, R_n, P_n are polynomials in ξ of degree $n-1$, of the form

$$U_n(\xi) = \sum_{k=2}^{n} U_{nk}\xi^{k-1}, \quad R_n(\xi) = \sum_{k=1}^{n} R_{nk}\xi^{k-1}, \quad P_n(\xi) = \sum_{k=1}^{n} P_{nk}\xi^{k-1}. \tag{14}$$

Substituting these into the differential equations and transferred shock-wave conditions, and equating like powers of ξ as well as t, yields for each approximation a system of $3n$ linear algebraic equations for the coefficients U_{nk}, R_{nk}, P_{nk} and X_n, whose non-homogeneous terms depend on all previous approximations.

Solving those equations in the third approximation gives for the position of the shock wave

$$\begin{aligned} X(t) &= \tfrac{1}{2}(\gamma+1)t + \frac{\gamma(\gamma+1)(\gamma-1)}{8(2\gamma-1)}jt^2 \\ &\quad + \frac{(\gamma+1)(\gamma-1)}{48(7\gamma-5)}\left[(\gamma+1)(3\gamma+1)j + \frac{\gamma(13\gamma^3-21\gamma^2+13\gamma-1)}{(2\gamma-1)^2}j^2\right]t^3 + \dots. \end{aligned} \tag{15}$$

This result is shown in figure 1 for a spherical shock wave with $\gamma = \frac{7}{5}$. Lee (1968) has carried the expansion to this order numerically for a cylindrical piston with $\gamma = \frac{7}{5}$. The second and third coefficients in (15) agree with his results to three significant figures.

In the special case of a spherical piston with $\gamma = 3$ we have carried the hand calculation to the fourth approximation analytically, which gives $X_4 = 9691/2850$, and to the fifth approximation numerically, giving $X_5 = 7{\cdot}24262901$. The two authors checked these tedious hand calculations, which were needed to check in turn and debug the computer programs, by carrying them out independently in the Northern and Southern hemispheres.

3. Extension of the series by computer

To carry the series further, we have independently written computer programs to calculate the general term. Both versions consist of some 400 Fortran statements.

The convective terms introduce quadruple summations into the non-homogeneous parts of the momentum and energy equations for U_{nk}, R_{nk} and P_{nk}. Consequently the program contains DO-loops nested six deep. These quickly become the most time-consuming part of the calculation, so the computing time for the nth approximation increases eventually as n^6. Thus on an IBM 3033 machine we computed 20 terms in 6 seconds and 40 terms in 280 seconds.

Although the coefficients, such as those in (15), are rational fractions when γ is rational, we compute them in floating-point arithmetic. Comparing the Australian and American results, and also double- and quadruple-precision computations,

n	Spherical, $\gamma = \frac{7}{5}$	Spherical, $\gamma = \frac{5}{3}$	Spherical, $\gamma = 3$	Cylindrical, $\gamma = \frac{7}{5}$
1	1·200000000000	1·333333333333	2·00000000000	1·2000000000000
2	0·186666666667	0·317460317460	1·20000000000	0·0933333333333
3	0·188345679012	0·330964978584	1·83333333333	0·0730864197531
4	0·172851981806	0·351087328915	3·40035087719	0·0577257959714
5	0·172147226896	0·428702976041	7·24262900585	0·0497185254748
6	0·195748089820	0·581262688522	16·7325356185	0·0473867537972
7	0·239592510180	0·833416073327	40·8212062145	0·0487020337051
8	0·303219524757	1·24182040572	103·538798073	0·0525457596193
9	0·394337922617	1·90667020627	270·351164204	0·0586078973893
10	0·525663995528	2·99573095341	721·973134446	0·0670385267585
11	0·714271423746	4·79335492559	1962·93555769	0·0782473038694
12	0·985060389731	7·78505460535	5415·71134591	0·0928536362648
13	1·37561449412	12·8028036868	15125·3041521	0·111712634840
14	1·94193338406	21·2785506061	42681·0787588	0·135973603154
15	2·76700088699	35·6880234991	121509·247882	0·167161791445
16	3·97437632751	60·3290517468	348589·799633	0·207289223128
17	5·74887230925	102·690500277	1006783·95686	0·259004706586
18	8·36757126135	175·866803349	2925043·77126	0·325795437783
19	12·2467590407	302·827404305	8543150·61409	0·412256433109
20	18·0133655273	523·983733067	25069946·9513	0·524449933038
21	26·6137638895	910·630204719	73881275·4824	0·670384889712
22	39·4795522908	1588·86850668	218567708·399	0·860657162621
23	58·7805913213	2782·27435391	648869068·945	1·10930506416
24	87·8118838110	4888·12883923	1932484742·18	1·43495376450
25	131·585889835	8613·85327622	5772286224·84	1·86234753012
26	197·740487538	15221·5900368	17288250591·8	2·42440315613
27	297·932522944	26967·3254176	51908194965·1	3·16496440419
28	449·979223858	47890·4406560	156214990411	4·14250004943
29	681·152558004	85235·2928220	471129305758	5·43507307638
30	1033·25274612	152014·101220	1·42371888519 12	7·14702352524
31	1570·42985951	271633·152889	4·31039960943 12	9·41796318002
32	2391·25395142	486253·291668	1·30727810033 13	12·4348912563
33	3647·35908070	871915·946437	3·97126124722 13	16·4485262673
34	5572·26775610	1565938·77695	1·20824798683 14	21·7953372274
35	8525·99348990	2816584·06448	3·68138936143 14	28·9272839088
36	13064·1157676	5073195·16260	1·12320678078 15	38·4519908447
37	20044·8405790	9149924·83352	3·43136709434 15	51·1870510605
38	30795·0631275	16523403·6091	1·04955212172 16	68·2334756229
39	47368·1399675	29874379·4238	3·21397456855 16	91·0751000913
40	72944·3025390	54074091·6579	9·85273540521 16	121·713200768

TABLE 1. Coefficients X_n in series (8) for shock wave

showed that less than half a significant figure is lost to truncation and round-off errors in each step. Thus in a double-precision calculation, starting with 16 significant figures, only five figures of agreement remain at $n = 25$. We therefore carried out all final computations in quadruple-precision arithmetic, starting with 31 figures. We are confident that this leaves even the 40th approximation correct to at least 14 figures.

We have computed four cases: a spherical piston with $\gamma = \frac{7}{5}, \frac{5}{3}$, and 3, and a cylindrical piston with $\gamma = \frac{7}{5}$. Although the history of the entire flow field is represented by our series, we concentrate on a single global quantity, the location of the shock

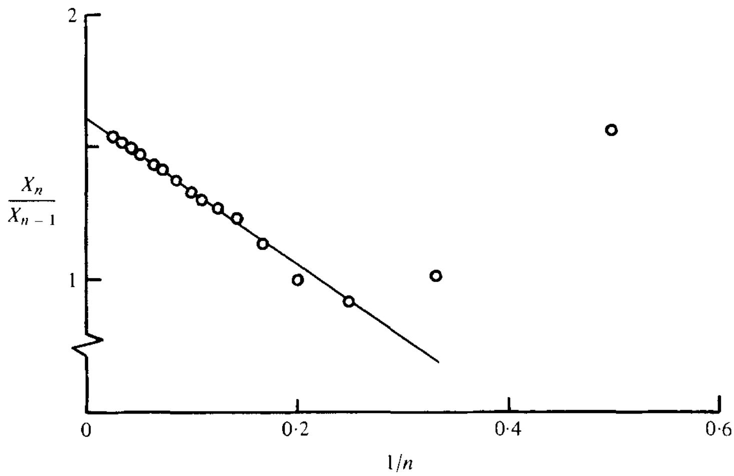

FIGURE 2. Graphical ratio test of Domb & Sykes for series (8) for position of shock wave. ——, $1{\cdot}61(1-1{\cdot}717/n)$.

wave. Table 1 lists the first 40 coefficients in its Taylor-series expansion (8), of which we have given the first three analytically in (15). For brevity we show them rounded to 12 figures, although in the subsequent analysis we have worked with at least 16 and where necessary all 31 figures.

4. Analysis of the coefficients for the shock wave

The coefficients are all positive, which means that the nearest singularity of the shock-wave function $X(t)$ lies on the positive real axis of t. We will verify that it is the Guderley singularity, corresponding to collapse of the shock wave onto the axis. The coefficients increase steadily in magnitude, indicating that the radius of convergence is less than unity, as it must be because the piston itself would reach the axis at $t = 1$. They increase faster for the spherical than the cylindrical piston, because the focusing is more intense, and faster as the adiabatic exponent γ increases, because according to the Newtonian theory of hypersonic flow the shock wave hugs the piston when $\gamma = 1$.

We can estimate the radius of convergence by plotting the ratios X_n/X_{n-1} of successive coefficients versus $1/n$ (Domb & Sykes 1957), for if the nearest singularity has the form

$$X(t) = \Sigma X_n t^n \sim A_1(1-t/t_c)^{\alpha_1} \quad \text{as} \quad t \to t_c \tag{16}$$

then

$$\frac{X_n}{X_{n-1}} \sim \frac{1}{t_c}\left(1-\frac{1+\alpha_1}{n}\right) \quad \text{as} \quad n \to \infty. \tag{17}$$

Figure 2 shows that for the spherical piston with $\gamma = \frac{7}{5}$ we can plausibly fit a linear asymptote using Guderley's exponent $\alpha_1 = 0{\cdot}717$. The intercept at $1{\cdot}61$ yields $t_c = 0{\cdot}62$ to graphical accuracy.

We refine this estimate of the radius of convergence by fitting polynomials in $1/n$, which is conveniently done by forming Neville tables (Gaunt & Guttmann 1974). Table 2 for the intercept shows that the result varies smoothly with n only in the first three columns. From these we estimate $1/t_c = 1{\cdot}609021 \pm 0{\cdot}000002$, so that

n	Linear fit	Quadratic	Cubic	Quartic	Quintic
35	1·53007606	1·60902299	1·60902320	1·60901781	1·60897478
36	1·53226903	1·60902297	1·60902268	1·60901702	1·60901071
37	1·53434346	1·60902294	1·60902240	1·60901919	1·60903714
38	1·53630871	1·60902290	1·60902220	1·60901988	1·60902569
39	1·53817317	1·60902286	1·60902199	1·60901948	1·60901596
40	1·53994441	1·60902280	1·60902180	1·60901946	1·60901928

TABLE 2. Bottom left-hand corner of Neville table for reciprocal radius of convergence $1/t_c$, for spherical piston with $\gamma = \frac{7}{5}$

n	Linear fit	Quadratic	Cubic	Quartic	Quintic
35	0·7172797	0·7172883	0·7169526	0·7134154	0·6695954
36	0·7172794	0·7172668	0·7169034	0·7163687	0·7396590
37	0·7172787	0·7172546	0·7170430	0·7186080	0·7368280
38	0·7172779	0·7172458	0·7170882	0·7176092	0·7092316
39	0·7172768	0·7172363	0·7170609	0·7167365	0·7091971
40	0·7172756	0·7172274	0·7170595	0·7170431	0·7197686

TABLE 3. Bottom left-hand corner of Neville table for exponent α_1, for spherical piston with $\gamma = \frac{7}{5}$

n	t_0	Linear extrapolation	Cubic	Quartic	Quintic
35	0·6282083504	0·6214859	0·621496081	0·6214960435	0·6214960443
36	0·6280216326	0·6214865	0·621496077	0·6214960448	0·6214960533
37	0·6278450217	0·6214870	0·621496073	0·6214960443	0·6214960406
38	0·6276777187	0·6214875	0·621496070	0·6214960432	0·6214960363
39	0·6275190066	0·6214879	0·621496067	0·6214960427	0·6214960395
40	0·6273682404	0·6214884	0·621496065	0·6214960425	0·6214960404

TABLE 4. Selected columns from bottom of Neville table for time for shock wave to reach centre, for spherical piston with $\gamma = \frac{7}{5}$

$t_c = 0{\cdot}621\,496 \pm 0{\cdot}000\,001$. Table 3 for the exponent is much more erratic, and we can only confirm Guderley's value to his three figures.

To verify that the nearest singularity corresponds to collapse of the shock wave onto the axis, we calculate the time t_0 for $X(t)$ to reach unity. In table 4 the second column shows the values obtained using n terms of the series (8) for the spherical shock wave with $\gamma = \frac{7}{5}$. These seem to be approaching our value of $t_c = 0{\cdot}621\,496$, and this is confirmed by forming a Neville table. In fact, table 4 is so much smoother than table 2 that it gives the radius of convergence to two more figures. For our other three cases the Neville tables are even better behaved. Thus we estimate with confidence

$$t_c = \begin{cases} 0{\cdot}62149\,604 & \text{for spherical piston with } \gamma = \frac{7}{5}, \\ 0{\cdot}52915\,2937 & \text{for spherical piston with } \gamma = \frac{5}{3}, \\ 0{\cdot}31284\,7355 & \text{for spherical piston with } \gamma = 3, \\ 0{\cdot}71394\,2146 & \text{for cylindrical piston with } \gamma = \frac{7}{5}, \end{cases} \tag{18}$$

with a possible error of one unit in the last figure.

Geometry	γ	α_1	α_2	α_3	A_1	A_2	A_3
Spherical	$\frac{7}{5}$	0·7171745	2·045	3·4	0·981706	0·0140	0·007
Spherical	$\frac{5}{3}$	0·6883768	1·885	3·1	0·989732	0·0055	0·006
Spherical	3	0·636411	1·638	2·5	1·016952	−0·0244	0·01
Cylindrical	$\frac{7}{5}$	0·835324	2·033	3	0·983865	0·0133	0·01

TABLE 5. Exponents and amplitudes in Guderley's local expansion (19)

5. Extraction of the local expansion

Our 40-term series will yield accurate results throughout the flow field almost up to the instant of collapse. We now show that the local singular behaviour in that vicinity can also be extracted from our coefficients with good accuracy.

Guderley (1942) conjectured that in the neighbourhood of collapse onto the axis the history of the shock wave is described more precisely by an expansion of the form (in our notation)

$$R(t) \equiv 1 - X(t) \sim \sum_{i=1} A_i(1-t/t_c)^{\alpha_i}. \tag{19}$$

He computed only the first exponent α_1, and the amplitudes A_i cannot be determined from local considerations. To extract this local expansion from our global solution we adopt a procedure devised by Baker & Hunter (1973) for estimating any number of exponents and corresponding amplitudes associated with a confluent singularity, given its location. We first form, from our series for $R(t)$, the expansion of an auxiliary function $\mathscr{R}$ that is meromorphic and therefore amenable to study by the method of Padé approximants.

In (19), using our estimates (18) for t_c, we rewrite the series (8) for $X(t)$ in powers of the auxiliary variable τ defined by $t = t_c[1-\exp(-\tau)]$, and then multiply the nth term by $n!$. Summing over n then yields the series for the auxiliary function

$$\mathscr{R}(\tau) = \sum_{i=1} \frac{A_i}{1+\alpha_i \tau}. \tag{20}$$

This has simple poles at $\tau = -1/\alpha_i$ and corresponding residues A_i/α_i. We evaluate these by forming Padé approximants (Baker 1965) to $\mathscr{R}(\tau)$.

The $[(N-1)/N]$ approximants, which are rational fractions with the denominator of degree one higher than the numerator, show the most consistent values for the poles and residues as N is varied up to our maximum possible value of 20. Thus we find the values listed in table 5 for the first three exponents and amplitudes in (19). We have checked most of these independently using Neville tables, and believe that they are correct to within one unit in the last figure given. For the first exponent this is confirmed by the values 0·717 174 50, 0·688 376 82, 0·636 410 60 and 0·835 323 20 given by Lazarus & Richtmyer (1977). Hunter & Baker (1973) have shown that the expected errors in the radius of convergence t_c, leading exponent α_1, and amplitude A_1 stand in the ratio $1:N:\ln N$, where N is the order of the last term in the series analysed. Our values in (18) and table 5 are in general agreement with this result.

	Spherical, $\gamma = \frac{7}{5}$	Spherical, $\gamma = \frac{5}{3}$	Spherical, $\gamma = 3$	Cylindrical, $\gamma = \frac{7}{5}$
Guderley (1942)	0·717	—	—	0·834
Butler (1954)	0·717173	0·688377	—	0·835217
Stanyukovich (1955)	0·717	—	0·638	0·834
Brushlinskii & Kazhdan (1963)	0·7170	0·68838	0·6364	—
Welsh (1967)	0·717174	0·688377	0·636411	0·835323
Goldman (1973)	—	0·688377	—	—
Lazarus & Richtmyer (1977)	0·71717450	0·68837682	0·63641060	0·83532320
Fujimoto & Mishkin (1978)	0·707	0·687	0·623	—
Mishkin & Fujimoto (1978)	—	—	—	0·828

TABLE 6. History of Guderley's similarity exponent α_1

6. Discussion

Although ours is a global solution, we have emphasized its singular behaviour near the instant of collapse, both because that is the most severe test, and because it represents the local problem that has been much studied since it was first considered by Guderley (1942).

Table 6 shows the history of values computed for Guderley's similarity exponent α_1 in our four cases. The values in table 5 that we have extracted from our 40-term series are seen to agree with all previous results from 1942 to 1977, and to be exceeded in accuracy only by the very precise calculations of Lazarus & Richtmyer.

We disagree, however, with the recent values of Fujimoto & Mishkin (1978) and Mishkin & Fujimoto (1978). We are indebted to Nelson Kemp for pointing out that the analytic solution proposed by those authors is just the approximation introduced by Stanyukovich (1955) in his equation (64; 80), which he characterizes as 'almost exact'.

Our determination of Guderley's local expansion turns out to yield remarkable accuracy – at least for the piston motions that we have considered – not only near the collapse but even at the start. We see from table 5 that over its whole course the radius of the converging shock wave is given correct to within 2 % by just the first term, and within less than $\frac{1}{2}$ % by the first three terms.

If the piston motion is altered – for example, to constant acceleration instead of constant speed – the time t_c for the shock wave to reach the axis and the amplitudes A_i in Guderley's local expansion (19) will all change, but the similarity exponent α_1 will remain unchanged. More precisely, this is certainly the situation when the adiabatic exponent γ of the gas is less than a critical value γ_c, which Brushlinskii & Kazhdan (1963) both calculate as 1·87 for a spherical shock wave. Lazarus & Richtmyer (1977) refine this to 1·8697680, and give $\gamma_c = 1{\cdot}9092084$ for cylindrical flow. Welsh suggests that there is no physical significance to this critical value, across which the transition is smooth, and that the local self-similar solution is unique for all γ. On the other hand, Brushlinkskii & Kazhdan (1963) report that in 1956 Kazhdan, Alalykin and Osserovich discovered that for any $\gamma > \gamma_c$ there exists a whole series of possible values for the similarity exponent α_1 that correspond to analytic solutions. According to Barenblatt (1979) Gel'fand has conjectured that the smallest value is always realized; and we see that this happens for our spherical piston with $\gamma = 3$. However, we presume that a piston motion could be contrived to realize any one of the higher values instead.

It is interesting to ask whether the subsequent exponents α_2, α_3, ... in the local expansion are also universal, depending on the adiabatic exponent of the gas and the number of space dimensions but independent of how the shock wave is produced. That question could be answered numerically by treating a different piston motion by our method, or better by perturbing Guderley's solution.

The values in table 5 have led us to conjecture that the exponents are equally spaced. For example, α_2 is greater than α_1 by 1·328 for the spherical geometry with $\gamma = \frac{7}{5}$, and if α_3 were greater again by the same amount it would equal 3·373, which is consistent with our estimate. The accuracy of our analysis allows just a hint of a fourth exponent, with a value between 4 and 5; and that too is consistent with equal spacing. Recently Lazarus (private communication) has confirmed analytically our conjecture of equally spaced exponents under certain plausible assumptions.

Often a singularity in any one flow quantity corresponds to singularities in all the others. Here, however, Joseph Keller has pointed out that the collapse of the shock wave will only at later times affect the body of the flow. To illustrate this, we have examined the series for the pressure on the surface of the piston. The Domb–Sykes plots are not as smooth as figure 2, but show a long-period oscillation, and Neville tables are correspondingly more erratic. However it is clear that for our spherical piston with $\gamma = \frac{7}{5}$ the nearest singularity lies at $t = 0{\cdot}90$, whereas the shock wave collapses at $t = 0{\cdot}62$. This singularity corresponds to the outgoing characteristic from the point of collapse, which would appear in the flow field except that the reflected shock wave appears first. It remains implicit in our series as a limiting line, an envelope of outgoing characteristics (Butler 1954).

The technique of computer-extended perturbation series used here is relatively new in fluid mechanics, and still under development. It is so far limited to simple geometries; and even then it has in some cases led to surprising and hence controversial results (e.g. Van Dyke 1978). However, the treatment of progressive water waves by Schwartz (1974) is generally regarded as a triumph of the method; and we believe that the present application demonstrates again that in favourable cases it yields results that can scarcely be obtained in any other way.

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